1.

Use the Laplace transform to solve the given initial value problem.

$$y^{(4)} - 81y = 0;\ y(0) = 17,\ y'(0) = 27,\ y''(0) = 81,\ y'''(0) = 135$$

$$L\{y^{(4)} - 81y\} = 0$$

$$s^4 L\{y\} - s^3 y(0) - s^2 y'(0) - s y''(0) - y'''(0) - 81 L\{y\} = 0$$

$$s^4 L\{y\} - 81 L\{y\} = 17s^3 + 27s^2 + 81s + 135$$

$$L\{y\} = \frac{17s^3 + 27s^2 + 81s + 135}{s^4 - 81}$$

$$L\{y\} = \frac{17s^3 + 27s^2 + 81s + 135}{(s^2+9)(s^2-9)}$$

$$\frac{As+B}{s^2+9} + \frac{Cs+D}{s^2-9} = \frac{17s^3 + 27s^2 + 81s + 135}{(s^2+9)(s^2-9)}$$

$$(As+B)(s^2-9) + (Cs+D)(s^2+9) = 17s^3 + 27s^2 + 81s + 135$$

$$As^3 - 9As + Bs^2 - 9B + Cs^3 + 9Cs + Ds^2 + 9D = 17s^3 + 27s^2 + 81s + 135$$

$$As^3 + Cs^3 = 17s^3 \implies A + C = 17$$

$$Bs^2 + Ds^2 = 27s^2 \implies B + D = 27$$

$$-9As + 9Cs = 81s \implies -9A + 9C = 81$$

$$-9B + 9D = 135 \implies -9B + 9D = 135$$

$$9[A + C = 17]$$
$$+\ -9A + 9C = 81$$
$$18C = 81 + 9(17)$$
$$C = \frac{234}{18} = 13$$
$$A = 4$$

$$9[B + D = 27]$$
$$-9B + 9D = 135$$
$$18D = 135 + 9(27)$$
$$D = 21$$
$$B = 6$$

$$Y(s) = \frac{4s+6}{s^2+9} + \frac{13s+21}{s^2-9} \longrightarrow \frac{A}{s+3} + \frac{B}{s-3} = \frac{13s+21}{(s+3)(s-3)}$$

$$A(s-3) + B(s+3) = 13s + 21$$

when $s = 3$ $B = 10$

when $s = -3$ $A = 3$

$$\frac{3}{s+3} + \frac{10}{s-3}$$
$$\downarrow$$
$$3e^{-3t} + 10e^{3t}$$

$$Y(t) = \frac{4s}{s^2+9} + \frac{6}{s^2+9} + \frac{13s}{s^2-9} + \frac{21}{s^2-9}$$

$$y(t) = 4\cos(3t) + 2\sin(3t) + 13\cosh(3t) + 7\sinh(3t)$$

or

$$y(t) = 4\cos(3t) + 2\sin(3t) + 3e^{-3t} + 10e^{3t}$$

2.

Find the inverse Laplace transform $\mathcal{L}^{-1}\{F(s)\}$ of the given function.

$$F(s) = \frac{16}{s^2+16}$$

Your answer should be a function of t.

$F(s) = \frac{16}{s^2+16}$ $\quad$ $L\{\sin(at)\} = \frac{a}{s^2+a^2}$

$L^{-1}\{F(s)\} = 4\sin(4t)$

3.

Use the Laplace transform to solve the given initial value problem.

$$y'' - 6y' - 112y = 0; \; y(0) = 6, \; y'(0) = 40$$

$L\{y'' - 6y' - 112y\} = 0$

$s^2 L\{y\} - sy(0) - y'(0) - 6[sL\{y\} - y(0)] - 112L\{y\} = 0$

$s^2 L\{y\} - 6sL\{y\} - 112L\{y\} = 6s + 40 - 36$

$(s^2 - 6s - 112)L\{y\} = 6s + 4$

$Y(s) = \dfrac{6s+4}{s^2 - 6s - 112}$

$Y(s) = \dfrac{6s+4}{(s+8)(s-14)}$ $\qquad \dfrac{A}{s+8} + \dfrac{B}{s-14} = \dfrac{6s+4}{(s+8)(s-14)}$

$A(s-14) + B(s+8) = 6s+4$

when $s = 14$ $\quad B = 4$

when $s = -8$ $\quad A = 2$

$\begin{array}{r} {}^{2}14 \\ \times\ 6 \\ \hline 84 \end{array}$ $\qquad \dfrac{88}{22}$

$Y(s) = \dfrac{2}{s+8} + \dfrac{4}{s-14}$

$L^{-1}\{Y(s)\} = 2e^{-8t} + 4e^{14t}$

$y(t) = 2e^{-8t} + 4e^{14t}$

4. Use the Laplace transform to solve the given initial value problem.

$$y' - y = e^{3t}, \quad y(0) = 2.$$

$L\{y' - y\} = L\{e^{3t}\}$

$sL\{y\} - y(0) - L\{y\} = \frac{1}{s-3}$

$(s-1)L\{y\} = \frac{1}{s-3} + 2$

$L\{y\} = \frac{1 + 2s - 6}{(s-3)(s-1)}$

$L\{y\} = \frac{2s-5}{(s-3)(s-1)}$

$\frac{A}{s-3} + \frac{B}{s-1} = \frac{2s-5}{(s-3)(s-1)}$

$A(s-1) + B(s-3) = 2s-5$

when $s = 1$ $B = \frac{3}{2}$

when $s = 3$ $A = \frac{1}{2}$

$y(s) = \frac{\frac{1}{2}}{s-3} + \frac{\frac{3}{2}}{s-1}$

$L^{-1}\{y(s)\} = \frac{1}{2}e^{3t} + \frac{3}{2}e^{t}$

$y(t) = \frac{1}{2}e^{3t} + \frac{3}{2}e^{t}$

5. Use the Laplace transform to solve the given initial value problem.

$$y'' - 3y' + 2y = e^{3t}, \quad y(0) = 1,\ y'(0) = 0.$$

$$L\{y'' - 3y' + 2y\} = L\{e^{3t}\}$$

$$s^2 L\{y\} - sy(0) - y'(0) - 3[sL\{y\} - y(0)] + 2L\{y\} = L\{e^{3t}\}$$

$$(s^2 - 3s + 2)L\{y\} - s(1) - (0) + 3(1) = \frac{1}{s-3}$$

$$(s^2 - 3s + 2)L\{y\} = \frac{1}{s-3} + s - 3$$

$$(s^2 - 3s + 2)L\{y\} = \frac{s^2 - 3s - 3s + 9 + 1}{s-3}$$

$$L\{y\} = \frac{s^2 - 6s + 10}{(s-3)(s-2)(s-1)}$$

$$\frac{A}{s-3} + \frac{B}{s-2} + \frac{C}{s-1} = \frac{s^2 - 6s + 10}{(s-3)(s-2)(s-1)}$$

$$A(s-2)(s-1) + B(s-3)(s-1) + C(s-3)(s-2) = s^2 - 6s + 10$$

when $s = 2$, $B = -2$

when $s = 1$, $C = \frac{5}{2}$

when $s = 3$, $A = \frac{1}{2}$

$$Y(s) = \frac{\frac{1}{2}}{s-3} + \frac{-2}{s-2} + \frac{\frac{5}{2}}{s-1}$$

$$L^{-1}\{Y(s)\} = \frac{1}{2}e^{3t} - 2e^{2t} + \frac{5}{2}e^{t}$$

$$y(t) = \frac{1}{2}e^{3t} - 2e^{2t} + \frac{5}{2}e^{t}$$

6. Use the Laplace transform to solve the given initial value problem.

$$y'' - 10y' + 9y = 5t, \quad y(0) = -1,\, y'(0) = 2.$$

$L\{y'' - 10y' + 9y\} = L\{5t\}$

$s^2 L\{y\} - sy(0) - y'(0) - 10[sL\{y\} - y(0)] + 9L\{y\} = 5L\{t\}$ $\qquad t^n = \frac{n!}{s^{n+1}}$

$(s^2 - 10s + 9)L\{y\} - s(-1) - (2) + 10(-1) = \frac{5}{s^2}$

$(s^2 - 10s + 9)L\{y\} = \frac{5}{s^2} - s + 2 + 10$

$(s^2 - 10s + 9)L\{y\} = \frac{5 - s^3 + 12s^2}{s^2}$

$L\{y\} = \frac{-s^3 + 12s^2 + 5}{s^2(s-1)(s-9)}$

$\frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-1} + \frac{D}{s-9} = \frac{-s^3 + 12s^2 + 5}{s^2(s-1)(s-9)}$

$As^2(s-1)(s-9) + B(s-1)(s-9)s + Cs^2(s-9)s + Ds^2(s-1)\cdot s = -s^3 + 12s^2 + 5$

when $s = 1$ $\quad C = -2$

when $s = 9$ $\quad D = \frac{31}{81}$

when $s = 0$ $\quad B = \frac{5}{9}$

$A = \frac{50}{81}$

$81(8)D = -81(9) + 12(81) + 5$

$8D = -9 + 12 + \frac{5}{81}$

$D = \frac{248}{81} \cdot \frac{1}{8} = \frac{31}{81}$ $\qquad 8\overline{)248}$ 31

$Y(s) = \frac{\frac{50}{81}}{s} + \frac{\frac{5}{9}}{s^2} + \frac{-2}{s-1} + \frac{\frac{31}{81}}{s-9}$

$L^{-1}\{Y(s)\} = \frac{50}{81} + \frac{5t}{9} - 2e^t + \frac{31}{81}e^{9t}$

$y(t) = \frac{50}{81} + \frac{5t}{9} - 2e^t + \frac{31}{81}e^{9t}$

7. Use the Laplace transform to solve the given initial value problem.

$$y'' - 6y' + 15y = 2\sin 3t, \quad y(0) = -1,\ y'(0) = -4.$$

$$L\{y'' - 6y' + 15y\} = L\{2\sin(3t)\}$$

$$s^2 L\{y\} - sy(0) - y'(0) - 6\left[sL\{y\} - y(0)\right] + 15L\{y\} = L\{2\sin(3t)\}$$

$$(s^2 - 6s + 15)L\{y\} - s(-1) - (-4) + 6(-1) = \frac{6}{s^2+9}$$

$$(s^2 - 6s + 15)L\{y\} = \frac{6 - s(s^2+9) + 2(s^2+9)}{s^2+9}$$

$$Y(s) = \frac{-s^3 + 2s^2 - 9s + 24}{(s^2+9)((s-3)^2+6)}$$

$$\frac{As+B}{(s^2+9)} + \frac{Cs+D}{((s-3)^2+6)} = \frac{-s^3 + 2s^2 - 9s + 24}{(s^2+9)((s-3)^2+6)}$$

$$(As+B)((s-3)^2+6) + (Cs+D)(s^2+9) = -s^3 + 2s^2 - 9s + 24$$

$$As^3 - 6As^2 + 15As + Bs^2 - 6Bs + 15B + Cs^3 + 9Cs + Ds^2 + 9D = -s^3 + 2s^2 - 9s + 24$$

$$As^3 + Cs^3 = -s^3 \Longrightarrow A + C = -1$$

$$-6As^2 + Bs^2 + Ds^2 = 2s^2 \Longrightarrow -6A + B + D = 2$$

$$15As - 6Bs + 9Cs = -9s \Longrightarrow 15A - 6B + 9C = -9$$

$$15B + 9D = 24$$

$$D = \frac{24 - 15B}{9}, \quad -6A + B + \frac{24-15B}{9} = 2$$

$$C = -A - 1, \quad 15A - 6B + 9(-A-1) = -9$$

$$6A - 6B = 0$$

$$B - 6B + \frac{24 - 15B}{9} = 2$$

$$-5B(9) + 24 - 15B = 18$$

$$-60B = -6$$

$$B = \frac{1}{10}$$

$$B = \frac{1}{10}$$

$$A = \frac{1}{10}$$

$$C = \frac{-11}{10}$$

$$D = \frac{25}{10} = \frac{5}{2}$$

$$Y(s) = \frac{\frac{1}{10}s + \frac{1}{10}}{s^2+9} + \frac{\frac{-11}{10}s + \frac{5}{2}}{((s-3)^2+6)}$$

$$Y(s) = \frac{\frac{1}{10}s}{s^2+9} + \frac{\frac{1}{10}}{s^2+9}$$

$$\frac{1}{10}\left[\frac{-11s+25}{((s-3)^2+6)}\right] = \frac{1}{10}\left[\frac{-11(s-3)-8}{((s-3)^2+6)}\right]$$

$$= \frac{-11}{10}\left[\frac{s-3}{((s-3)^2+6)}\right] - \frac{8}{\sqrt{6}}\left[\frac{\sqrt{6}}{((s-3)^2+6}\right]$$

$$L^{-1}\{Y(s)\} = \frac{1}{10}\cos(3t) + \frac{1}{30}\sin(3t) - \frac{11}{10}e^{3t}\cos(\sqrt{6}t) - \frac{8}{10\sqrt{6}}e^{3t}\sin(\sqrt{6}t)$$

$$\boxed{Y(t) = \frac{1}{10}\cos(3t) + \frac{1}{30}\sin(3t) - \frac{11}{10}e^{3t}\cos(\sqrt{6}t) - \frac{8}{10\sqrt{6}}e^{3t}\sin(\sqrt{6}t)}$$